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The spherical limit of the n-vector model and correlation inequalities

N Angelescu, M Bundaru and G Costache
Institute of Physics and Nuclear Engineering, Bucharest, PO Box 5206, Romania

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Abstract. The asymptotics of the state of the n-vector model with a finite number of spins in the spherical limit is studied. Besides re-deriving the limit of the free energy, corresponding to a generalised spherical model (with 'spherical constraint' at every site), we also obtain the limit of the correlation functions, which allows a precise definition of the state of the latter model. Correlation inequalities are proved for ferromagnetic interactions in the asymptotic regime. In particular, it is shown that the generalised spherical model fulfils the expected Griffiths-type inequalities, differing in this respect from the spherical model with overall constraint.

1. Introduction

A powerful method for studying models of statistical mechanics consists of the exploitation of correlation inequalities. Its fruitfulness has been fully revealed in the case of ferromagnetic Ising models, for which a large variety of such inequalities is available (see e.g. Griffiths 1972, Sylvester 1976 and references therein). In the last few years, a great deal of consideration has been given to deriving analogous inequalities for ferromagnetic classical n-vector models with $n > 1$. Ginibre (1970) settled the $n = 2$ case, while recently Dunlop (1976) and Kunz *et al* (1976) have been able to exploit the cases $n = 1, 2$ to provide, for $n = 3, 4$, besides Griffiths-type inequalities for correlations involving only one spin component, new inequalities for correlations between different spin components. However, even for $n = 3, 4$, the proof of some natural correlation inequalities is still an open problem. As the only result known for all n seems to be that of Pearce and Thompson (1976) on the Ising-like n-vector model, while the isotropic n-vector model is completely unexplored for $n > 4$, and in view of the difficulties encountered in such a study, it should be interesting to see, at least, to what extent these (and what other kind of) inequalities are valid in the large-n limit. It is principally to this latter problem that we devote this paper. Naturally we are lead to consider the asymptotic study of the spherical ($n \to \infty$) limit of the isotropic n-vector model in an external magnetic field, not only for the free energy (which, in fact, has been already done by Knops (1973) and Pearce and Thompson (1977), who obtain, as limit of the free energy per spin component, the free energy of the generalised spherical model of Bettoney and Mazo (1970)), but also for the correlation functions. This is known as $1/n$ expansion for the correlation functions. The idea originates in the paper by Abe (1973), but the only rigorous result in this direction (concerning, however, only the limit of the magnetisation) seems to be contained in the paper by Pearce and Thompson (1977).

Our main result is that the expected Griffiths-type inequalities, including the concavity of the magnetisation, hold in the asymptotic region. This shows that the generalised spherical model has the expected normal behaviour, in contrast to the usual spherical model of Berlin and Kac (1952) for which anomalies appear (see e.g. Barber and Fisher 1973, Pearce 1976). Particular cases of Griffiths inequalities for the generalised spherical models of films have previously been obtained by Costache (1977), who makes a detailed study of the thermodynamics of the model as well.

The paper is organised as follows. In § 2 the models under consideration (n-vector, Gaussian and spherical) are defined, the notation made precise, and a suitable representation of the n-vector partition function given. Section 3 contains the asymptotic study of the n-vector correlations. Some technical details and the proof of an analytic version of the steepest-descents method, which is the core of the argument for the asymptotic development, are relegated to appendixes 1–3. In § 4 the correlation inequalities are proved on the basis of a result on M-matrices, which might present some interest in itself.

2. The models

We shall describe here the models under consideration. They all consist of a finite number of 'spins', labelled by an italic letter (i, j, k or l), taking on integer values $1, 2, \ldots, N$. Their number, N, will be held fixed throughout the paper.

Real- or complex-valued functions of the spin index i, considered as vectors in R^N or C^N will be denoted by boldface letters $\boldsymbol{h} = (h_1, \ldots, h_N)$. C^N is given the usual scalar product: $(\boldsymbol{h}, \boldsymbol{h}') = \Sigma_{i=1}^N h_i h_i'$. $M_N(R)(M_N(C))$ is the space of $N \times N$ real (complex) matrices, considered as operators on C^N. We shall denote $M_N^0(R)(M_N^0(C))$ the subspace of those $J \in M_N(R)(M_N(C))$ satisfying

$$J_{ii} = 0, \qquad J_{ij} = J_{ji}, \qquad i, j = 1, 2, \ldots, N, \tag{2.1}$$

and choose $\{J_{ij}; 1 \leq i < j \leq N\}$ as coordinates on M_N^0.

Functions of the spin index i with values in $R^n(C^n)$, considered as vectors in $R^{Nn}(C^{Nn})$, will be denoted by boldface letters with an arrow, $\vec{\boldsymbol{\xi}} = (\vec{\xi}_1, \ldots, \vec{\xi}_N)$. The coordinates of $\vec{\xi}_i \in R^n$ will be labelled by superscript Greek indices $\mu, \nu, \ldots$. For our purposes it will be convenient to view $\vec{\boldsymbol{\xi}}$ as a collection of n elements of C^N, $\boldsymbol{\xi}^\mu$, $\mu = 1, \ldots, n$, where ξ_i^μ is the μ coordinate of $\vec{\xi}_i$.

2.1. *The isotropic n-vector model*

The phase space is

$$S_n = \{\vec{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}^\mu)_{\mu=1,\ldots,n} \in R^{Nn} \,\Big|\, \sum_{\mu=1}^n (\sigma_i^\mu)^2 = n, i = 1, \ldots, N\}$$

(i.e. all configurations of N n-dimensional 'spins' of length $n^{1/2}$), with its natural measure $d\mu_n(\vec{\boldsymbol{\sigma}})$ induced by Lebesgue measure $d\vec{\boldsymbol{\sigma}}$. The interaction energy is

$$\mathscr{H}(\vec{\boldsymbol{\sigma}}; J, \vec{\boldsymbol{\xi}}) = -\sum_{\mu=1}^n [\tfrac{1}{2}(J\boldsymbol{\sigma}^\mu, \boldsymbol{\sigma}^\mu) + (\boldsymbol{\xi}^\mu, \boldsymbol{\sigma}^\mu)], \tag{2.2}$$

where the interaction matrix $J \in M_N^0(R)$, and $\vec{\boldsymbol{\xi}} \in R^{Nn}$ is a distribution of magnetic fields

$\vec{\xi}_i$ at every spin site i. The partition function is

$$Z_n(J, \vec{\xi}) = \int_{S_n} \exp(-\beta\mathcal{H}(\vec{\sigma}; J, \vec{\xi}))\, d\mu_n(\vec{\sigma}) \qquad (\beta = (k_B T)^{-1}). \tag{2.3}$$

Occasionally, we shall make J and $\vec{\xi}$ complex (maintaining relations (2.1)). Thereby, clearly, $Z_n(J, \vec{\xi})$ is analytic and does not vanish on $M_N^0(R) \times R^{Nn}$.

We shall in fact be concerned only with a particular kind of configuration of magnetic fields: all $\vec{\xi}_i$ along the same direction, namely the diagonal of $R^n(C^n)$. We agree to denote throughout the paper by $\vec{h}$ such a distribution of magnetic fields, and by $\boldsymbol{h}$ the common value of all $\boldsymbol{h}^\mu$.

In this case the free energy will be denoted

$$F_n(J, \boldsymbol{h}) \equiv -\beta^{-1} \lg Z_n(J, \vec{h}) \tag{2.4}$$

and will be an analytic function on $M_N^0(R) \times R^N$.

The equilibrium state of the model, characterised by $\beta, J, \vec{h}$, is defined by the sequence of all the truncated correlation functions

$$\left\langle \prod_{\mu\in\Omega} \left(\prod_{i \in A_\mu} \sigma_i^\mu \right) \right\rangle^{(n)\mathrm{T}} = \left[\prod_{\mu\in\Omega} \left(\prod_{i\in A_\mu} \beta^{-1} \frac{\partial}{\partial \xi_i^\mu} \right) \right] \lg Z_n(J; \vec{h} + \vec{\xi}) \Big|_{\vec{\xi}=0}. \tag{2.5}$$

Here Ω is an arbitrary subset of component indices, and A_μ are finite non-void collections of site indices. The usual (non-truncated) correlation functions will be denoted without the superscript 'T'.

Owing to the choice of $\vec{h}$, the n-vector model Hamiltonian (2.2) is invariant under the permutation of spin components. This is reflected in the correlation functions as well, e.g. $\langle \sigma_i^\mu \rangle^{(n)}$ is independent of μ, $\langle \sigma_i^\mu \sigma_j^\nu \rangle^{(n)}$ takes on only two values for any given i, j, depending on whether $\mu = \nu$ or $\mu \neq \nu$, and so on.

Special attention will be paid to those combinations of correlation functions which appear as derivatives of the free energy per spin component $(1/n)F_n(J, \boldsymbol{h})$, such as the local magnetisation

$$m_i^{(n)} = -\frac{1}{n} \frac{\partial F_n(J, \boldsymbol{h})}{\partial h_i} = \frac{1}{n} \sum_{\mu=1}^{n} \langle \sigma_i^\mu \rangle^{(n)} = \langle \sigma_i^\mu \rangle^{(n)} \tag{2.6}$$

and the local susceptibilities

$$\chi_{ij}^{(n)} = \frac{\partial m_i^{(n)}}{\partial h_j} = \frac{\beta}{n} \sum_{\mu,\nu=1}^{n} \langle \sigma_i^\mu \sigma_j^\nu \rangle^{(n)\mathrm{T}}. \tag{2.7}$$

2.2. The Gaussian model

The phase space is R^N (i.e. all configurations of N one-dimensional 'spins' of arbitrary length) with Lebesgue measure $d\boldsymbol{\sigma}$. For a given interaction matrix $J \in M_N^0(R)$, we define

$$D_J = \{\boldsymbol{\gamma} \in R^N | X \equiv \Gamma - J > 0, \Gamma_{ij} = \gamma_i \delta_{ij}\}. \tag{2.8}$$

The partition function, for a positive definite $X \in M_N(R)$, a magnetic field distribution $\boldsymbol{h} \in R^N$ and an inverse temperature β, is

$$Q(X, \boldsymbol{h}) = \int_{R^N} \exp\{-\beta[\tfrac{1}{2}(X\boldsymbol{\sigma}, \boldsymbol{\sigma}) - (\boldsymbol{h}, \boldsymbol{\sigma}) - \tfrac{1}{2} \operatorname{tr} X]\}\, d\boldsymbol{\sigma}. \tag{2.9}$$

It will be convenient to write X as $\Gamma - J$, with $J \in M_N^0(R)$ and Γ a diagonal matrix with diagonal vector $\boldsymbol{\gamma} \in D_J$.

Again X and $\boldsymbol{h}$ can be made complex, and $Q(X, \boldsymbol{h})$ will thereby be analytic on $\{X \in M_N(C) | X_{ij} = X_{ji}, \operatorname{Re} X > 0\} \times C^N$ and given by

$$Q(X, \boldsymbol{h}) = \det(\beta X/2\pi)^{-1/2} \exp\{(\beta/2)[(X^{-1}\boldsymbol{h}, \bar{\boldsymbol{h}}) + \operatorname{tr} X]\}. \tag{2.9'}$$

The truncated correlation functions which define the equilibrium state are

$$\left\langle \prod_{i \in A} \sigma_i \right\rangle^{\mathrm{T}} = \left(\prod_{i \in A} \beta^{-1} \frac{\partial}{\partial h_i} \right) \lg Q(X, \boldsymbol{h}), \tag{2.10}$$

where A is a finite sequence of site indices.

2.3. *The generalised spherical model*

This model is defined from the Gaussian model above by making a certain choice of $\boldsymbol{\gamma}$ depending on J and $\boldsymbol{h}$: $\boldsymbol{\gamma}$ is to be determined from the following system of equations,

$$(X^{-1})_{ii} = \beta[1 - (X^{-1}\boldsymbol{h})_i^2], \qquad i = 1, 2, \ldots, N, \tag{2.11}$$

which can be viewed either as a minimum condition for $Q(X, \boldsymbol{h})$ considered as a function of $\boldsymbol{\gamma} \in D_J$, or as a set of ('individual spherical') constraints on the random vector $\boldsymbol{\sigma} \in R^N$:

$$\langle \sigma_i^2 \rangle = 1, \qquad i = 1, \ldots, N. \tag{2.11'}$$

The system (2.11) has a unique solution: $\boldsymbol{\gamma}^0(J, \boldsymbol{h}) \in D_J$.

Indeed, $Q(X, \boldsymbol{h})$ is a strictly log-convex function of $\boldsymbol{\gamma}$ on the convex domain D_J. This can be checked directly on the Hessian matrix

$$H_{ij}(X, \boldsymbol{h}) \equiv \partial^2 \lg Q(X, \boldsymbol{h})/\partial\gamma_i\, \partial\gamma_j = \tfrac{1}{2}(X^{-1})_{ij}^2 + \beta(X^{-1}\boldsymbol{h})_i (X^{-1})_{ij} (X^{-1}\boldsymbol{h})_j, \tag{2.12}$$

which is obviously strictly positive definite for $\boldsymbol{\gamma} \in D_J$. (The Schur product of strictly positive definite matrices is strictly positive definite.) Moreover, $\lim \lg Q(X, \boldsymbol{h}) = \infty$ for $\boldsymbol{\gamma} \to \partial D_J$ or $\|\boldsymbol{\gamma}\| \to \infty$.

J and h can be made complex as above; the implicit function theorem and the continuity of H assure then that: There exists a complex neighbourhood of $M_N^0(R) \times R^N$ on which the solution $\boldsymbol{\gamma}^0(J, \boldsymbol{h})$ defined above extends analytically and $\operatorname{Re} H > 0$.

From now on we agree to denote always by $\boldsymbol{\gamma}^0$ this solution of the system (2.11), by Γ_0 the diagonal matrix with diagonal vector $\boldsymbol{\gamma}^0$, by $\langle \ldots \rangle_0^{\mathrm{T}}$ the correlation functions of the Gaussian model (2.10) for $X = X_0 \equiv \Gamma_0 - J$, and by $Q_0(J, \boldsymbol{h})$ the Gaussian model partition function at $X = X_0(J, \boldsymbol{h})$, viewed as a function of $(J, \boldsymbol{h})$.

The generalised spherical model is thus defined by postulating the following expression for its free energy:

$$F^0(J, \boldsymbol{h}) = -\beta^{-1} \lg Q_0(J, \boldsymbol{h}). \tag{2.13}$$

We shall see that the appropriate definitions of the local magnetisations and susceptibilities are respectively

$$m_i^0(J, \boldsymbol{h}) = -\partial F^0(J, \boldsymbol{h})/\partial h_i = (X_0^{-1}\boldsymbol{h})_i, \tag{2.14}$$

$$\chi_{ij}^0(J, \boldsymbol{h}) = \partial m_i^0(J, \boldsymbol{h})/\partial h_j = (X_0^{-1})_{ij} - \beta(X_0^{-1} M_0 H_0^{-1} M_0 X_0^{-1})_{ij}, \tag{2.15}$$

where M_0 is the diagonal matrix with diagonal vector $\boldsymbol{m}^0$, and $H_0 = H(X_0, \boldsymbol{h})$. In obtaining (2.14) and (2.15) use has been made of the minimum condition (2.11).

We conclude this section by giving a certain representation of $Z_n(J, \vec{\boldsymbol{\xi}})$, which proves to be very useful in studying the behaviour of the n-vector model when n becomes large. This representation has already been used in a similar context (Stanley 1968, Abe 1973, Joyce 1973 and references therein). We shall, however, include a proof, to provide the missing details necessary to make rigorous the usual derivation by means of the δ-function representation.

For every $J \in M_N^0(C)$, $\vec{\boldsymbol{\xi}} \in C^{Nn}$ and $n \geq 3$:

$$Z_n(J, \vec{\boldsymbol{\xi}}) = \left(\frac{\beta n^{1/2}}{2\pi}\right)^N \int_{\boldsymbol{R}^N} \prod_{\mu=1}^{n} Q(X + \mathrm{i}T, \xi^\mu)\,\mathrm{d}\boldsymbol{t}, \tag{2.16}$$

where T is the diagonal matrix with diagonal $\boldsymbol{t}$, and $\gamma \in C^N$ is arbitrary as long as $\operatorname{Re} X > 0$.

Proof. For $\boldsymbol{\rho} \in \boldsymbol{R}_+^N$ let

$$S_{\boldsymbol{\rho}} = \left\{ \vec{\boldsymbol{\sigma}} \in \boldsymbol{R}^{Nn} \,\middle|\, \sum_{\mu=1}^{n} (\sigma_i^\mu)^2 = \rho_i,\ i = 1, \ldots, N \right\}.$$

Let Φ: $\boldsymbol{R}^N \to C$ be defined by

$$\Phi(\boldsymbol{\rho}) = \begin{cases} \left(\prod_{i=1}^{N} n\rho_i^{-1} \exp[-\beta\gamma_i(\rho_i - n)]\right)^{1/2} \int_{S_{\boldsymbol{\rho}}} \exp[-\beta H(\vec{\boldsymbol{\sigma}}; J, \vec{\boldsymbol{\xi}})]\,\mathrm{d}\mu_{\boldsymbol{\rho}}(\vec{\boldsymbol{\sigma}}), & \text{if } \boldsymbol{\rho} \in \boldsymbol{R}_+^N \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathrm{d}\mu_{\boldsymbol{\rho}}(\vec{\boldsymbol{\sigma}})$ is the natural measure on $S_{\boldsymbol{\rho}}$. Clearly, $\Phi(n, \ldots, n) = Z_n(J, \vec{\boldsymbol{\xi}})$. $\Phi(\boldsymbol{\rho})$ is continuous and integrable, and its Fourier transform $\hat{\Phi}(\boldsymbol{t})$ is also integrable for $n \geq 3$. The latter fact can be seen by calculating $\hat{\Phi}$ explicitly and by majorising $|\hat{\Phi}|$, using the inequalities in appendices 1 and 2; however, for $n \geq 5$, the following simple argument can be used: as $\Phi(\boldsymbol{\rho}) \sim \rho_i^{(n-2)/2}$ for $\rho_i \searrow 0$, we have

$$\left[\prod_{i=1}^{N} \left(1 - \frac{\partial^2}{\partial \rho_i^2}\right)\right] \Phi(\boldsymbol{\rho}) \in L_1(\boldsymbol{R}^N),$$

so

$$|\hat{\Phi}(\boldsymbol{t})| \leq \text{constant} \times \prod_{i=1}^{N} (1 + t_i^2)^{-1} \in L_1(\boldsymbol{R}^N).$$

Applying a well-known result on the inversion of the Fourier transform (Stein and Weiss 1971), one can write

$$\Phi(n, \ldots, n) = (2\pi)^{-N/2} \int_{\boldsymbol{R}^N} \exp\left(\mathrm{i}n \sum_{i=1}^{N} t_i\right) \hat{\Phi}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t},$$

which leads, after a few calculations, to (2.16).

3. The asymptotics of the isotropic n-vector model for large n

The purpose of this section is to give a detailed description of the state of the n-vector model when $n \to \infty$. As already stated, we have chosen to formulate the results for 'equally distributed' magnetic fields $\vec{h} = (h^\mu)$: $h^\mu = h$, $\forall\mu$. It should be stressed that a definite *ansatz* concerning the direction and strength of the magnetic fields as n varies should by all means be done, in order to give sense to the $n \to \infty$ limit; the picture depends strongly, though in a foreseen way, on this *ansatz*.

The results are as follows:

(1) For every fixed finite set Ω of component indices, the truncated correlation functions $\langle \Pi_{\mu\in\Omega} (\Pi_{i\in A_\mu} \sigma_i^\mu)\rangle^{(n)\mathrm{T}}$ (where A_μ are arbitrary non-void finite collections of site indices) have asymptotic series in powers of n^{-1} as $n \to \infty$. In particular, their $n \to \infty$ limit exists and equals $\langle \Pi_{i\in A_\mu} \sigma_i\rangle_0^{\mathrm{T}}$ if Ω consists of only one element, μ, and zero otherwise.

For further reference we shall also write down the n^{-1} corrections for the simplest correlation functions:

$$\langle \sigma_i^\mu\rangle^{(n)} = \langle\sigma_i\rangle_0 - (2n\beta)^{-1}(\partial/\partial h_i) \lg\det H_0 + \mathrm{O}(n^{-2}), \tag{3.1}$$

$$\langle \sigma_i^\mu \sigma_j^\nu\rangle^{(n)\mathrm{T}} = \delta_{\mu\nu}\langle\sigma_i\sigma_j\rangle_0 - (n\beta^2)^{-1}\left[\delta_{\mu\nu}\frac{\partial^2}{\partial h_i\,\partial h_j}\lg\det H_0 + \left(H_0\frac{\partial\gamma^0}{\partial h_i}, \frac{\partial\gamma^0}{\partial h_j}\right)\right] + \mathrm{O}(n^{-2}), \tag{3.2}$$

$$\langle \sigma_i^\mu\sigma_j^\nu\sigma_k^\rho\rangle^{(n)\mathrm{T}} = (n\beta^2)^{-1}\left(\delta_{\mu\nu}\frac{\partial}{\partial h_k}(X_0^{-1})_{ij} + \delta_{\nu\rho}\frac{\partial}{\partial h_i}(X_0^{-1})_{jk} + \delta_{\rho\mu}\frac{\partial}{\partial h_j}(X_0^{-1})_{ki}\right) + \mathrm{O}(n^{-2}). \tag{3.3}$$

(2) The free energy per spin component $(1/n)F_n(J, h)$ has an asymptotic series in powers of n^{-1} for every $(J, h) \in M_N^0(R)\times R^N$. Explicitly, to order n^{-2}

$$(1/n)F_n(J,h) = F^0(J,h) + (2\beta n)^{-1}\lg\det(2\pi\beta^{-2}H_0) + \mathrm{O}(n^{-2}). \tag{3.4}$$

Term-by-term derivatives with respect to J, h of this series yield asymptotic series for the corresponding correlation functions of the n-vector model. For instance, putting aside $m_i^{(n)}$ for which (3.1) is again obtained, one has

$$\chi_{ij}^{(n)} = \chi_{ij}^0 - (2\beta n)^{-1}\frac{\partial^2}{\partial h_i\partial h_j}\lg\det H_0 + \mathrm{O}(n^{-2}), \tag{3.5}$$

$$\frac{\partial m_k^{(n)}}{\partial J_{ij}} = \frac{\partial m_k^0}{\partial J_{ij}} - (2\beta n)^{-1}\frac{\partial^2}{\partial h_k\partial J_{ij}}\lg\det H_0 + \mathrm{O}(n^{-2}), \tag{3.6}$$

$$\frac{\partial^2 m_k^{(n)}}{\partial h_i\partial h_j} = \frac{\partial^2 m_k^0}{\partial h_i\partial h_j} - (2\beta n)^{-1}\frac{\partial^3}{\partial h_i\partial h_j\partial h_k}\lg\det H_0 + \mathrm{O}(n^{-2}), \tag{3.7}$$

where the definitions (2.6), (2.7), (2.14), (2.15) have been used.

One can see from result (1) that, loosely speaking, the state of the n-vector model converges to the product state of an infinite number of copies of Gaussian models satisfying the minimum condition (2.11). These copies of Gaussian models are coupled only in higher-order corrections; it is not too difficult to see that the first non-zero term in the asymptotic series of $\langle \Pi_{\mu\in\Omega} (\Pi_{i\in A_\mu}\sigma_i^\mu)\rangle^{(n)\mathrm{T}}$ is at most of the order $n^{1-|\Omega|}$, where $|\Omega|$ is the number of points of Ω.

Result (2) is meant to make precise the sense in which the n-vector model converges to a generalised spherical model. Namely, certain 'mean values' of correlation functions are selected (those which can be obtained through derivations of the free energy with respect to h and J) which converge in the limit; their limits are the corresponding derivatives of the generalised spherical model free energy and can be used to define the 'correlation functions' (or state) of that model. Result (2) should be contained in (1), and in fact is, but there is no easy way of showing this explicitly, because higher-order terms of the asymptotic series in (1) contribute non-vanishingly to a certain fixed-order term in the asymptotic series in (2).

Thus our description of the spherical limit of the n-vector model (with a fixed number of spins) is far more complete than that given in previous approaches (Abe 1973, Knops 1973, Pearce and Thompson 1977), in that all n-vector correlations to arbitrary order in n^{-1} are studied in the limit, and the state of the spherical model is well defined in terms of these. It is perhaps useful, before starting the formal proof, to outline the general argument providing the above results.

Both statements can be formulated as the convergence of a sequence of functions and of all their derivatives. The functions in question can be extended analytically to a common complex domain, and this kind of convergence will follow from uniform convergence on this domain. Uniform convergence follows in turn from an analytic version of the steepest-descents method, which, for the reader's convenience, is proved in appendix 3.

Proof of (1). According to (2.5), the truncated correlation functions can be obtained by taking logarithmic derivatives with respect to $\vec{\xi}$ of $\lg G_n(\vec{\xi})$ at $\vec{\xi}=0$, where

$$G_n(\vec{\xi}) = Z_n(J, \vec{h}+\vec{\xi})(Q(X_0, \boldsymbol{h}))^{-n+|\Omega|} \tag{3.8}$$

is an entire function in $\boldsymbol{\xi}^\mu \in C^N$, $\mu \in \Omega$. Using the representation (2.16) we have

$$G_n(\vec{\xi}) = \left(\frac{\beta n^{1/2}}{2\pi}\right)^N \int_{R^N} \left(\prod_{\mu\in\Omega} Q(X_0+\mathrm{i}T, \boldsymbol{h}+\boldsymbol{\xi}^\mu)\right)\left(\frac{Q(X_0+\mathrm{i}T, \boldsymbol{h})}{Q(X_0, \boldsymbol{h})}\right)^{n-|\Omega|} \mathrm{d}\boldsymbol{t}. \tag{3.9}$$

Taking advantage of the fact that $Q(X_0+\mathrm{i}T, \boldsymbol{h}+\boldsymbol{\xi})$ is bounded away from zero as a function of $\boldsymbol{t}\in R^N$, as seen from its analytic form (2.9′) and using the estimates (A1.2) and (A2.1), we introduce the following notations,

$$\begin{aligned} f(\boldsymbol{t}) &= \lg\,(Q(X_0+\mathrm{i}T, \boldsymbol{h})/Q(X_0, \boldsymbol{h})), \\ g(\boldsymbol{t}, \vec{\xi}) &= \sum_{\mu\in\Omega} \lg Q(X_0+\mathrm{i}T, \boldsymbol{h}+\boldsymbol{\xi}^\mu) - |\Omega| f(\boldsymbol{t}), \end{aligned} \tag{3.10}$$

in terms of which

$$G_n(\vec{\xi}) = \left(\frac{\beta n^{1/2}}{2\pi}\right)^N \int_{R^N} \exp(nf(\boldsymbol{t}) + g(\boldsymbol{t}, \vec{\xi}))\,\mathrm{d}\boldsymbol{t}. \tag{3.9′}$$

This is already of the form entering in the proposition in appendix 3, with f, g having the required analyticity properties. Clearly, $f(0)=0$, $\partial_i f(0)=0$ by the minimum condition (2.11), and $-\partial_i\partial_j f(0) = (H_0)_{ij}$ (given by equation (2.12) at $X=X_0$) is positive definite. As Re g is bounded above as a function of $\boldsymbol{t}\in R^N$ uniformly for $\vec{\xi}$ in compacts of $C^{|\Omega|N}$, we are left only to show that, with D_n defined in (A3.1), $\int_{R^N\setminus D_n} |\exp nf(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}$ falls off more rapidly than any inverse power of n. Indeed, taking into account that X_0 is self-adjoint, inequality (A2.1) (written for $Y=X_0^{-1}$) gives

$$\left|\frac{\det X_0}{\det(X_0+\mathrm{i}T)}\right|^{n/2} \leqslant \prod_{j=1}^{N} (1+a^2 t_j^2)^{-n/4},$$

where $a = \|X_0\|^{-1}$, while inequality (A1.1) gives

$$|\exp\{\beta/2[(X_0+\mathrm{i}T)^{-1}-X_0^{-1}]\boldsymbol{h}, \boldsymbol{h}\}|\leq 1,$$

from which

$$\begin{aligned}\int_{R^N\backslash D_n} |\exp nf(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t} &\leq \int_{R^N\backslash D_n} \prod_{i=1}^{N} (1+a^2t_i^2)^{-n/4}\,\mathrm{d}\boldsymbol{t} \\ &\leq 2N\left(\int_{-\infty}^{\infty} (1+a^2t^2)^{-n/4}\,\mathrm{d}t\right)^{N-1} \int_{n^{-1/2}\lg n}^{\infty} (1+a^2t^2)^{-n/4}\,\mathrm{d}t \\ &\leq C_N \exp[-a^2(\lg n)^2].\end{aligned}$$

Thus, by the proposition in appendix 3, we conclude that $G_n(\vec{\boldsymbol{\xi}})$ has an asymptotic series uniformly for $\vec{\boldsymbol{\xi}}$ in compacts of $C^{|\Omega|N}$. As the first term of this series,

$$\lim_{n\to\infty} G_n(\vec{\boldsymbol{\xi}}) = \left(\frac{\beta}{2\pi}\right)^N \left(\prod_{\mu\in\Omega} Q(X_0, \boldsymbol{h}+\boldsymbol{\xi}^\mu)\right) \det\left(\frac{1}{2\pi}H_0\right)^{-1/2}, \tag{3.11}$$

is bounded away from zero, we can conclude that $\lg G_n$ has in turn an asymptotic series uniformly in $\vec{\boldsymbol{\xi}}$. Consequently, all its derivatives at $\vec{\boldsymbol{\xi}}=0$ have asymptotic series obtained by term-by-term derivation of the series of $\lg G_n$. In particular, the limit of truncated correlation functions follows from the logarithm of the function (3.11) and has the form given in the statement. Straightforward calculations provide also formulae (3.1)–(3.3).

Proof of (2). As remarked in § 2, $Q_0(J, \boldsymbol{h})$ extends analytically and has no zeros in a certain complex neighbourhood of $M_N^0(R)\times R^N$. Thus

$$\psi_n(J, \boldsymbol{h}) \equiv Z_n(J, \vec{\boldsymbol{h}})Q_0(J, \boldsymbol{h})^{-n} \tag{3.12}$$

are analytic functions of J, $\boldsymbol{h}$ in this neighbourhood for all n.

Define

$$f(\boldsymbol{t}; J, \boldsymbol{h}) = \lg(Q(X_0+\mathrm{i}T, \boldsymbol{h})/Q(X_0, \boldsymbol{h})) \tag{3.13}$$

which is analytic in $(\boldsymbol{t}, J, \boldsymbol{h})$ on a complex neighbourhood of $R^N\times M_N^0(R)\times R^N$, because, in view of (A1.2) and (A2.1), $Q(X_0+\mathrm{i}T, \boldsymbol{h})/Q(X_0, \boldsymbol{h})$ is bounded away from zero, and satisfies $f(0; J, \boldsymbol{h})\equiv 0$, $\partial_i f(0; J, \boldsymbol{h})\equiv 0$. Thus

$$\psi_n(J, \boldsymbol{h}) = \left(\frac{\beta n^{1/2}}{2\pi}\right)^N \int_{R^N} \exp(nf(\boldsymbol{t}; J, \boldsymbol{h}))\,\mathrm{d}\boldsymbol{t}, \tag{3.14}$$

and the result (2) will follow from appendix 3, provided we can show that uniformly for $(J, \boldsymbol{h})$ in a compact neighbourhood of every point in $M_N^0(R)\times R^N$

$$n^p \int_{R^N\backslash D_n} |\exp(nf(\boldsymbol{t}; J, \boldsymbol{h}))|\,\mathrm{d}\boldsymbol{t} \underset{n\to\infty}{\to} 0, \qquad p=1, 2, \ldots. \tag{3.15}$$

To this aim, choose a neighbourhood V of the given point in $M_N^0(R)\times R^N$ such that $\operatorname{Re} H(X_0, \boldsymbol{h})\geq c>0$ on V_1. Then, one can choose $\epsilon>0$ such that $\operatorname{Re} H(X_0+\mathrm{i}T, \boldsymbol{h})\geq c/2$ for $\{|t_i|<\epsilon, i=1, \ldots, N\}\equiv D(\epsilon)$. For n sufficiently large, $D_n\subset D(\epsilon)$, and we shall divide the integration domain as $(R^N \backslash D(\epsilon)) \cup (D(\epsilon)\backslash D_n)$. For $t \in D(\epsilon)\backslash D_n$, making use of

$$f(t; J, h) = -\int_0^1 \sigma \, d\sigma \int_0^1 d\tau (H(X_0 + i\sigma\tau T, h)t, t),$$

we have

$$\int_{D(\epsilon)\backslash D_n} |\exp(nf(t; J, h))| \, dt \leq \int_{R^N\backslash D_n} \exp\left(-\frac{nc}{4}\|t\|^2\right) dt \leq C_N \exp\left(-\frac{c}{4}(\lg n)^2\right).$$

Outside D_ϵ, at least one $|t_i|$ exceeds ϵ. Using the results in appendices 1 and 2,

$$\int_{R^N\backslash D(\epsilon)} |\exp(nf(t; J, h))| \, dt \leq K^n \int_\epsilon^\infty dt(\mu + \alpha^2 t^2)^{-n/4} \leq K'^n,$$

with $K \searrow 1$ for $\|\mathrm{Im}\, h\| \to 0$ and $\|\mathrm{Im}\, X_0\| \to 0$, and where one can choose $\mu \nearrow 1$ and α bounded away from zero in the same limit. Thus one can determine a neighbourhood $V_2 \subset V_1$ on which $K' < 1$, and consequently (3.15) is fulfilled.

From now on the proof proceeds along the same lines as for (1).

4. Correlation inequalities in the spherical limit

In this section we shall restrict ourselves to considering ferromagnetic interactions and prove in this case that certain correlation inequalities hold in the large-n limit of the n-vector model. Such a study might suggest, when combined with known results for small n (commented upon in § 1), what kind of correlation functions of the n-vector model can be expected to satisfy Griffiths-type inequalities for all n.

Let us first state precisely the assumptions:

(a) Isotropic, 'strictly' ferromagnetic interactions, i.e.

$$J \in M_N^0(R), \qquad J_{ij} \geq 0, \qquad (e^J)_{ij} > 0, \qquad i, j = 1, \ldots, N. \tag{4.1}$$

The last condition means that every two spins i, j feel each other, i.e. there is a sequence $i = i_0, i_1, \ldots, i_p, i_{p+1} = j$, such that $J_{i_k i_{k+1}} > 0$ for $k = 0, \ldots, p$. This is by no means a restriction of the generality, as otherwise the system decouples into non-interacting subsystems.

(b) Positive, 'equally distributed' magnetic fields, i.e.

$$h^\mu = h \in R^N, \qquad h_i \geq 0, \qquad i = 1, \ldots, N. \tag{4.2}$$

The results are as follows:

(3) The n-vector correlation functions containing only one spin component satisfy, for n sufficiently large, the inequalities

$$\left\langle \prod_{i \in A} \sigma_i^\mu \right\rangle^{(n)} \geq 0, \tag{4.3}$$

$$\left\langle \prod_{i \in A} \sigma_i^\mu \prod_{j \in B} \sigma_j^\mu \right\rangle^{(n)} - \left\langle \prod_{i \in A} \sigma_i^\mu \right\rangle^{(n)} \left\langle \prod_{j \in B} \sigma_j^\mu \right\rangle^{(n)} \geq 0, \tag{4.4}$$

$$\langle \sigma_i^\mu \sigma_j^\mu \sigma_k^\mu \rangle^{(n)\mathrm{T}} \leq 0. \tag{4.5}$$

On the other hand, for $\mu \neq \nu$, card A and card $B \leq 2$ and n sufficiently large,

$$\left\langle \prod_{i\in A} \sigma_i^\mu \prod_{j\in B} \sigma_j^\nu \right\rangle^{(n)} - \left\langle \prod_{i\in A} \sigma_i^\mu \right\rangle^{(n)} \left\langle \prod_{j\in B} \sigma_j^\nu \right\rangle^{(n)} \leq 0. \tag{4.6}$$

However, if card A = card $B = 2$, we are able to prove (4.6) only for sufficiently small $\boldsymbol{h}$.

(4) The n-vector correlation functions obtained as derivatives of $(1/n)F_n(J, \boldsymbol{h})$ satisfy, for n sufficiently large, the inequalities

$$m_i^{(n)} = \frac{1}{n} \sum_\mu \langle \sigma_i^\mu \rangle^{(n)} \geq 0, \tag{4.7}$$

$$\chi_{ij}^{(n)} = \frac{\beta}{n} \sum_{\mu,\nu=1}^{n} \langle \sigma_i^\mu \sigma_j^\nu \rangle^{(n)\mathrm{T}} > 0, \tag{4.8}$$

$$\frac{\partial m_k^{(n)}}{\partial J_{ij}} = \frac{\beta}{n} \sum_{\mu=1}^{n} (\langle \vec{\sigma}_i \vec{\sigma}_j \sigma_k^\mu \rangle^{(n)} - \langle \vec{\sigma}_i \vec{\sigma}_j \rangle^{(n)} \langle \sigma_k^\mu \rangle^{(n)}) \geq 0, \tag{4.9}$$

$$\frac{\partial^2 m_k^{(n)}}{\partial h_i \partial h_j} = \frac{\beta^2}{n} \sum_{\mu,\nu,\rho=1}^{n} \langle \sigma_i^\mu \sigma_j^\nu \sigma_k^\rho \rangle^{(n)\mathrm{T}} \leq 0, \tag{4.10}$$

$$\frac{\partial}{\partial J_{k1}} \langle \vec{\sigma}_i \vec{\sigma}_j \rangle^{(n)} = \beta (\langle (\vec{\sigma}_i \vec{\sigma}_j)(\vec{\sigma}_k \vec{\sigma}_1) \rangle^{(n)} - \langle \vec{\sigma}_i \vec{\sigma}_j \rangle^{(n)} \langle \vec{\sigma}_k \vec{\sigma}_l \rangle) > 0, \tag{4.11}$$

where again (4.11) is proved only for small $\boldsymbol{h}$.

Inequalities (4.3), (4.7) and (4.4), (4.8), (4.9), (4.11) are natural generalisations of the first and second GKS inequalities, while (4.5), (4.10) are related to the concavity of the magnetisation. All these inequalities, except for (4.5), (4.10), (4.11), are known to hold for $n \leq 4$ (cf Dunlop 1976, Kunz *et al* 1976 and references therein). As far as we know, (4.5) is known only for $n = 1$ (Griffiths *et al* 1970), and (4.11) only for $n = 1, 2$ (Ginibre 1970). Inequalities (4.3), (4.4) are also known for the extremely anisotropic n-vector model with arbitrary n (Pearce and Thompson 1976).

The proof of (3) and (4) will consist of showing that certain matrices have positive entries. At this point, the argument relies heavily on the concept of the M-matrix, of which, for the reader's convenience, we shall recall some equivalent definitions and derive certain related propositions (which could be interesting by themselves and might possibly present some novelty).

Let $\mathscr{A} \subset M_N(R)$ be the class of matrices having non-positive off-diagonal entries. Equivalently, $\mathscr{A} = \{A \in M_N(R) | (\mathrm{e}^{-tA})_{ij} \geq 0, \forall i, j = 1, \ldots, N, \forall t > 0\}$.

Definition. $A \in \mathscr{A}$ is called an M-matrix if it satisfies one of the following equivalent conditions:

(i) A is non-singular and A^{-1} has non-negative entries.
(ii) There exists $\boldsymbol{\xi} \in \boldsymbol{R}^N$ with $\xi_i > 0$, $i = 1, \ldots, N$, such that $(A\boldsymbol{\xi})_i > 0$, $i = 1, \ldots, N$.
(iii) $(\mathrm{e}^{-tA})_{ij}$ are integrable functions of t on $[0, \infty)$, $i, j = 1, \ldots, N$.
(iv) $(\mathrm{e}^{-tA})_{ij}$ converges to zero as $t \to \infty$, $i, j = 1, \ldots, N$.

It is immediately obvious that (i) implies (ii) and (iii) implies (iv). Making use of the matrix identity

$$1 - \mathrm{e}^{-tA} = A \int_0^t \mathrm{e}^{-\tau A} \, \mathrm{d}\tau = \int_0^t \mathrm{e}^{-\tau A} \, \mathrm{d}\tau \, . \, A \tag{4.12}$$

one can see that from (iv) it follows that $A\int_0^t e^{-\tau A}\, d\tau$ is non-singular, so A itself is non-singular. Letting $t\to\infty$ in (4.12) one obtains $A^{-1}=\int_0^\infty e^{-\tau A}\, d\tau$, which has non-negative entries. Thus (iv) implies (i). To see that (ii) implies (iii) one applies (4.12) to the vector $\boldsymbol{\xi}$. As $(e^{-tA}\boldsymbol{\xi})_i>0$ and $\int_0^t (e^{-\tau A}A\xi)_i\, d\tau$ are increasing functions of t, it follows that $(e^{-\tau A}\boldsymbol{\xi})_i$ are integrable functions on $[0,\infty]$. As $(A\boldsymbol{\xi})_i>0$ for all $i=1,\dots,N$, (iii) is proved.

As for self-adjoint A, (iii) is implied by the positive definiteness of A; we have as an immediate consequence that, under conditions (4.1), $X=\Gamma-J$ is an M-matrix as soon as $\gamma\in D_J$. In particular,

$$(X^{-1})_{ij}>0, \qquad i,j=1,\dots,N. \tag{4.13}$$

Lemma. Let $X\in M_N(R)$ be a self-adjoint M-matrix satisfying (4.13). Then

(a) The matrix Q defined by

$$Q_{ij}=\sum_{k=1}^{N} X_{ik}\frac{1}{(X^{-1}\boldsymbol{h})_k}(X^{-1})^2_{kj}, \qquad i,j=1,\dots,N, \tag{4.14}$$

where $\boldsymbol{h}\neq 0$ is an arbitrary vector with $h_i\geqslant 0$, $i=1,\dots,N$, is an M-matrix. Moreover, $(Q^{-1})_{ij}>0$ for all i,j, unless $\boldsymbol{h}$ is of the form $h_k=a\delta_{ks}$ for some s, in which case $(Q^{-1})_{is}=0$, $(Q^{-1})_{ki}>0$ for all $i\neq s$, and all k.

(b) The matrix P defined by

$$(P^{-1})_{ij}=(X^{-1})^2_{ij}, \qquad i,j=1,\dots,N \tag{4.15}$$

is an M-matrix.

Proof. (*a*) We shall firstly show that $Q\in\mathscr{A}$. Indeed, using Schwartz's inequality

$$\sum_{k\neq i}|X_{ik}|\frac{1}{\sum_{l=1}^N X^{-1}_{kl}h_l}(X^{-1})^2_{kj}\geqslant\frac{(\sum_{k\neq i}X^{-1}_{kj}|X_{ki}|)^2}{\sum_{k\neq i}|X_{ki}|\sum_{l\neq 1}^N X^{-1}_{kl}h_l}$$

and the fact that $\sum_{k\neq i}X^{-1}_{jk}|X_{ki}|=X^{-1}_{ij}X_{ii}-\delta_{ij}$, one arrives at

$$\sum_{k\neq i}|X_{ik}|\frac{1}{\sum_{l=1}^N X^{-1}_{kl}h_l}(X^{-1})^2_{kj}\geqslant\frac{(X^{-1}_{ij})^2X^2_{ii}}{X_{ii}\sum_{l=1}^N X^{-1}_{il}h_l-h_i}\geqslant(X^{-1}_{ij})^2\frac{1}{\sum_{l=1}^N X^{-1}_{il}h_l}X_{ii},$$

which shows that $Q\in\mathscr{A}$. Next we have to find a vector $\boldsymbol{\xi}$, with $\xi_i>0$, $i=1,\dots,N$, for which $(Q\boldsymbol{\xi})_i>0$, $i=1,\dots,N$. That such a vector exists there is evident owing to X being an M-matrix. Finally, $Q_{ij}\neq 0$ for all $i\neq j$ (implying $(Q^{-1})_{ij}>0$, as seen from $Q^{-1}=\int_0^\infty \exp(-\tau Q)\, d\tau$) if at least one of the inequalities above is strict. Indeed $Q_{is}=0$ implies (from having equality in Schwartz inequality) that

$$\sum_{l=1}^{N}(X^{-1})_{kl}(h_l-a\delta_{ls})=b\delta_{ki} \qquad \text{for some } a, b,$$

whence $h_l=a\delta_{ls}+bX_{li}$, but also $h_i=0$, whence $b=0$. In this case, $Q_{ks}=0$ for all $k\neq s$, and $Q_{kj}<0$ for all $k\neq j$, $j\neq s$, which leads as above to the statement.

(*b*) The equivalence (i)–(ii) of the above definition of an M-matrix implies that Q^{-1} has only non-negative entries. Choosing in particular $h_i=\delta_{il}$, $i=1,\dots,N$, it means that, for any $i, l, k = 1, 2, \ldots, N$, $\Sigma_j P_{ij}X_{jl}^{-l}X_{kj}^{-1} \geq 0$. Multiplying this inequality by $X_{kl}(k \neq l)$ and summing over k gives

$$P_{il}X_{ll}^{-1} - \delta_{il}X_{ll} \leq 0,$$

showing that P is an M-matrix (recall that P^{-1} has only positive entries).

We are now prepared to prove results (3) and (4). In view of (1) and (2), these will follow from the corresponding inequalities for the first non-zero term of the asymptotic series. For simplicity, we shall assume that at least two of the components of $\boldsymbol{h}$ are non-zero; there is, however, sufficient information about Q^{-1} in point (a) of the lemma to cover the case of one non-vanishing component also.

Proof of (*3*). Inequalities (4.3) and (4.4) for the Gaussian model are proved in Leff (1971), the main ingredient being supplied by (4.13). The LHS of (4.5) has an asymptotic series starting in the order n^{-1}, given by (3.3), if $\boldsymbol{h} \neq 0$, and vanishes identically if $\boldsymbol{h} = 0$. As

$$\frac{\partial}{\partial h_k}(X_0^{-1})_{ij} = -\sum_{l=1}^{N} (X_0^{-1})_{il}\frac{\partial \gamma_l^0}{\partial h_k}(X_0^{-1})_{lj},$$

we have only to show that, for all $l, k = 1, \ldots, N$ and $\boldsymbol{h} \neq 0$,

$$\frac{\partial \gamma_l^0}{\partial h_k} = \beta(H_0^{-1}M_0X_0^{-1})_{lk} = \beta[(X_0M_0^{-1}H_0)^{-1}]_{lk} = \beta[(\tfrac{1}{2}Q_0 + \beta M_0)^{-1}]_{lk} > 0, \tag{4.16}$$

where Q_0 is defined by (4.14) in terms of X_0. This is obvious, as Q_0 is an M-matrix by the lemma and M_0 is diagonal and with positive diagonal.

Inequality (4.6) contains three cases, in all of them the asymptotic series starting with n^{-1} terms if $\boldsymbol{h} \neq 0$:

(i) Card A = card B = 1; this follows from (3.2), (4.16) and $(H_0)_{ij} > 0$.

(ii) Card A = 2, card B = 1; this follows by expressing the LHS in terms of truncated correlation functions, from (3.3) and the former case.

In both cases (i) and (ii), if $\boldsymbol{h} = 0$ the LHS vanishes identically for $n \geq 3$, owing to the rotational invariance. This is not the case, however, for case (iii), where in fact we shall give the proof only for $\boldsymbol{h} = 0$.

(iii) Card A = card B = 2, $\boldsymbol{h} = 0$:

$$\begin{aligned}\langle\sigma_{i_1}^{\mu}\sigma_{i_2}^{\mu}\sigma_{j_1}^{\nu}\sigma_{j_2}^{\nu}\rangle^{(n)} &- \langle\sigma_{i_1}^{\mu}s_{i_2}^{\mu}\rangle^{(n)}\langle\sigma_{j_1}^{\nu}\sigma_{j_2}^{\nu}\rangle^{(n)}|_{\boldsymbol{h}=0}\\
&= \langle\sigma_{i_1}^{\mu}\sigma_{i_2}^{\mu}\sigma_{j_1}^{\nu}\sigma_{j_2}^{\nu}\rangle^{(n)\mathrm{T}}|_{\boldsymbol{h}=0}\\
&= -\frac{1}{2n\beta^2}\sum_{k,l=1}^{N}(X_0^{-1})_{i_1k}(X_0^{-1})_{i_2k}(H_0^{-1})_{kl}(X_0^{-1})_{lj_1}(X_0^{-1})_{lj_2}|_{\boldsymbol{h}=0}.\end{aligned}$$

As for $\boldsymbol{h} = 0$, $(H_0)_{kl} = \frac{1}{2}(X_0^{-1})_{kl}^2$, point ($a$) of the lemma ensures that the sum over l is positive for all k, j_1, j_2, which finishes the proof.

Proof of (*4*). (4.7) is a particular case of (4.3). The LHS of (4.8) converges to χ_{ij}^0, equation (2.15), which can be put into the form

$$\chi_{ij}^0 = [(X_0 + 2\beta M_0P_0M_0)^{-1}]_{ij}, \tag{4.17}$$

where we denote P_0^{-1} the Schur square of X_0^{-1}, on which the inequality is obvious, as P_0 is an M-matrix by point (b) of the lemma. In proving (4.9) one can again suppose $\boldsymbol{h} \neq 0$ (because for $\boldsymbol{h}=0$, $\boldsymbol{m}^{(n)}=0$ for all J), and the limiting value is

$$\frac{\partial m_k^0}{\partial J_{ij}}=\frac{1}{2}\sum_l\left((X_0^{-1})_{il}\sqrt{\frac{m_j^0}{m_i^0}}-(X_0^{-1})_{jl}\sqrt{\frac{m_i^0}{m_j^0}}\right)^2[(X_0M_0^{-1}H_0)^{-1}]_{lk}. \tag{4.18}$$

As, by the lemma, $X_0M_0^{-1}H_0$ is an M-matrix, (4.18) is positive.†

The limit of (4.10) is (consider again only $\boldsymbol{h}\neq 0$)

$$\partial\chi_{ij}^0/\partial h_k=-[\chi^0(\partial(\chi^0)^{-1}/\partial h_k)\chi^0]_{ij},$$

and, because $\partial\gamma_i^0/\partial h_k>0$, we have only to prove that

$$[\chi^0M_0(\partial P_0/\partial h_k)M_0\chi^0]_{ij}=-[\chi^0M_0P_0(\partial(P_0^{-1}/\partial h_k)P_0M_0\chi^0]_{ij}>0.$$

But $P_0M_0\chi^0$ has positive elements by the lemma (and equation (4.17)), and $\partial P_0^{-1}/\partial h_k$ has negative elements by explicit calculation.

The limit of (4.11) for $\boldsymbol{h}=0$ is

$$\begin{aligned}
\frac{1}{\beta}\frac{\partial}{\partial J_{kl}}&(\chi_0^{-1})_{ij}\\
&=\frac{1}{\beta}\bigg((X_0^{-1})_{ik}(X_0^{-1})_{lj}+(X_0^{-1})_{il}(X_0^{-1})_{kj}\\
&\quad-\sum_{p,q=1}^{N}(X_0^{-1})_{ip}(X_0^{-1})_{jp}(H_0^{-1})_{pq}(X_0^{-1})_{qk}(X_0^{-1})_{ql}\bigg)\\
&=\frac{1}{2\beta}\sum_{p=1}^{N}\left((X_0^{-1})_{ip}\sqrt{\frac{(X_0^{-1})_{jl}}{(X_0^{-1})_{il}}}-(X_0^{-1})_{jp}\sqrt{\frac{(X_0^{-1})_{il}}{(X_0^{-1})_{jl}}}\right)^2\\
&\quad\times\left(\sum_{q=1}^{N}(H_0^{-1})_{pq}(X_0^{-1})_{qk}(X_0^{-1})_{ql}\right),
\end{aligned}$$

which is positive by the same argument as for (4.18).

5. Concluding remarks

We have obtained, in § 3, asymptotic developments of the n-vector correlations for finite systems. We leave open the problem of the validity of a similar development in the thermodynamic limit. Much caution is necessary when trying to extend our results to infinite systems. Indeed, the proof of result (1) shows that an asymptotic expansion holds also for

$$\left[\prod_{\mu\in\Omega}\left(\prod_{i\in A_\mu}\beta^{-1}\frac{\partial}{\partial\xi_i^\mu}\right)\right]\lg Z_n(J;\vec{h}+\vec{\xi}),$$

without imposing $\vec{\xi}=0$ in the end. Thus, for instance, taking for simplicity $\vec{h}=0$ and $\xi_i^\mu=\xi\delta_{\mu 1}$, the magnetisation of the n-vector model will converge for $n\to\infty$ to

† If, however, $h_k=a\delta_{ks}$, $\partial m_s^0/\partial J_{ij}=0$. But also $\partial m_s^{(n)}/\partial J_{ij}=0$ in this case, because $f(\vec{\sigma}_s)=\int d\mu_n(\vec{\sigma}')\exp(-\beta\mathcal{H}(\vec{\sigma},J;0))$, where $\vec{\sigma}=(\vec{\sigma}',\vec{\sigma}_s)$, is independent of $\vec{\sigma}_s$ by rotational invariance.

$N^{-1}\xi\,\Sigma_{i,j=1}^{N}\,(X_0^{-1})_{ij}$, where γ_0 is determined from the constraint $(X_0^{-1})_{ii}=\beta$, which is independent of ξ. Choosing J translationally invariant, one can perform the thermodynamic limit of the limiting magnetisation, and this will eventually turn out to be infinite for β greater than β_c of the spherical model, while for $\beta<\beta_c$ it will be a linear function of ξ, leading as expected to spontaneous magnetisation equal to zero. On the other hand, there exist proofs (Pearce and Thompson 1977) showing that the n-vector magnetisation converges for n, $N\to\infty$ to the thermodynamic limit of $N^{-1}\,\Sigma_{i=1}^{N}\,m_i^0$. This shows the difficulties to be expected when studying the asymptotics of the state of the n-vector model for n, $N\to\infty$ simultaneously. Technically, these should be reflected by the fact that the thermodynamic limit generally shrinks the analytical domain of $G_n(\bar{\boldsymbol{\xi}})$, and one cannot expect uniform convergence as analytic functions to hold. Thus one can at most hope that result (2) might be extended to the thermodynamic limit of correlation functions.

Griffiths-type inequalities have been proved, in § 4, for the n-vector correlations in the large-n limit. In particular, it has been shown that the generalised spherical model obeys these inequalities. This is not true for the usual spherical model with overall constraint, where a counter-example is available (Pearce 1976).

It would be interesting to check on the asymptotic series in § 4 whether certain correlation functions are monotonic functions of n, at least in the large-n region, which in turn would imply monotonic convergence of the critical temperature of the n-vector model to that of a spherical model. This could be accomplished by looking at the next term of the asymptotic series, which is somewhat more difficult and will be left for further consideration. Let us only remark here that such monotonicity properties have already been looked for by Stanley (1969), and partial results, though for a differently scaled n-vector model, obtained; e.g. inequalities between $n=1$ and 2 or between these and an arbitrary n (cf Thompson 1973, Bricmont 1976, Kunz *et al* 1976), or the monotonicity of the two-point function for the periodic one-dimensional model (cf Milosevic *et al* 1970).

Appendix 1

Let $X=X_1+\mathrm{i}X_2\in M_N(C)$ with X_1, X_2 self-adjoint and $X_1>0$. Let $T\in M_N(C)$ be self-adjoint and $\boldsymbol{h}=\boldsymbol{h}_1+\mathrm{i}\boldsymbol{h}_2\in C^N$ with $\boldsymbol{h}_1$, $\boldsymbol{h}_2$ real. Then

$$\operatorname{Re}\{[(X+\mathrm{i}T)^{-1}-X^{-1}]\boldsymbol{h},\bar{\boldsymbol{h}}\}\leqslant 4\|\boldsymbol{h}\|\,\|X_1^{-1}\|\,\|\boldsymbol{h}_2\|+\|\boldsymbol{h}\|^2\|X_1^{-1}\|^3\|X_2\|^2. \tag{A1.1}$$

Proof. $\bar{\boldsymbol{h}}=\boldsymbol{h}-2\mathrm{i}\boldsymbol{h}_2$, so

$$\begin{aligned}\operatorname{Re}\{[(X+\mathrm{i}T)^{-1}-X^{-1}]\boldsymbol{h},\bar{\boldsymbol{h}}\}\\ &=[\![\{\operatorname{Re}[(X+\mathrm{i}T)^{-1}-X^{-1}]\}\boldsymbol{h},\boldsymbol{h}]\!]+2\operatorname{Im}\{[(X+\mathrm{i}T)^{-1}-X^{-1}]\boldsymbol{h},\boldsymbol{h}_2\}\\ &\leqslant\|\boldsymbol{h}\|^2\|\operatorname{Re}[(X+\mathrm{i}T)^{-1}-X^{-1}]\|+2\|\boldsymbol{h}\|\,\|(X+\mathrm{i}T)^{-1}-X^{-1}\|\,\|\boldsymbol{h}_2\|.\end{aligned}$$

Now

$$\begin{aligned}(X+\mathrm{i}T)^{-1}-X^{-1}=X_1^{-1/2}\{&[1+\mathrm{i}X_1^{-1/2}(X_2+T)X_1^{-1/2}]^{-1}\\ &-(1+\mathrm{i}X_1^{-1/2}X_2X_1^{-1/2})^{-1}\}X_1^{-1/2},\end{aligned}$$

so, majorising the norm of the sum with the sum of the norms, and using $\|(1+\mathrm{i}A)^{-1}\|\leqslant 1$

for self-adjoint A, we have

$$\|(X+\mathrm{i}T)^{-1}-X^{-1}\|\leqslant 2\|X_1^{-1/2}\|^2=2\|X_1^{-1}\|. \tag{A1.2}$$

On the other hand, as $\mathrm{Re}(1+\mathrm{i}A)^{-1}=(1+A^2)^{-1}=1-A^2(1+A^2)^{-1}$ for self-adjoint A, we have

$$\begin{aligned}
&\mathrm{Re}[(X+\mathrm{i}T)^{-1}-X^{-1}]\\
&\quad=X_1^{-1/2}[\![(X_1^{-1/2}X_2X_1^{-1/2})^2[1+(X_1^{-1/2}X_2X_1^{-1/2})^2]^{-1}-[X_1^{-1/2}(X_2+T)X_1^{-1/2}\\
&\qquad\times\{1+[X_1^{-1/2}(X_2+T)X_1^{-1/2}]^2\}^{-1}]\!]X_1^{-1/2}\\
&\quad\leqslant X_1^{-1}X_2X_1^{-1}X_2X_1^{-1}\\
&\quad\leqslant\|X_1^{-1}\|^3\|X_2\|^2,
\end{aligned}$$

whence (A1.1) follows.

Appendix 2

For $a, b>0$, let

$$\mathcal{M}_{a,b}=\{Y=Y_1+\mathrm{i}Y_2\in M_N(C)|\,Y_1=Y_1^*\geqslant a,\ Y_2=Y_2^*,\ \|Y_2\|\leqslant b\}.$$

For every $0\leqslant\mu\leqslant a^2/(a^2+b^2)$ and every $\alpha^2\leqslant a^2-\mu b^2/(1-\mu)$, the following inequality holds for $Y\in\mathcal{M}_{a,b}$ and $T=T^*\in M_N(C)$:

$$|\det(1+\mathrm{i}TY)|\geqslant\det(\mu+\alpha^2T^2)^{1/2}. \tag{A2.1}$$

Proof. Using $\det(1+BA)=\det(1+AB)$, $A, B\in M_N(C)$, we can write $\det(1+\mathrm{i}TY)=\det[1+Y_1^{1/2}TY_1^{1/2}(\mathrm{i}-Y_1^{-1/2}Y_2Y_1^{-1/2})]$. Thus, for $1<\lambda<1+a^2/b^2$, we have

$$\begin{aligned}
&|\det(1+\mathrm{i}TY)|^2\\
&\quad=\det[1+Y_1^{1/2}T(Y_1+Y_2Y_1^{-1}Y_2)TY_1^{1/2}-2\,\mathrm{Re}(Y_1^{1/2}TY_2Y_1^{-1/2})]\\
&\quad\geqslant\det\{1-\lambda^{-1}+Y_1^{1/2}T[Y_1-(\lambda-1)Y_2Y_1^{-1}Y_2]TY_1^{1/2}\}\\
&\quad\geqslant\det[\![1-\lambda^{-1}+\{a-[(\lambda-1)/a]\|Y_2\|^2\}Y_1^{1/2}T^2Y_1^{1/2}]\!]\\
&\quad=\det[\![1-\lambda^{-1}+\{a-[(\lambda-1)/a]\|Y_2\|^2\}TY_1T]\!]\\
&\quad\geqslant\det\{1-\lambda^{-1}+[a^2-(\lambda-1)\|Y_2\|^2]T^2\},
\end{aligned}$$

where the monotonicity of det on positive matrices and the matrix inequality $2\,\mathrm{Re}\,A\leqslant\lambda^{-1}+\lambda AA^*$ have been used.

Appendix 3

Let $V\subset C^M$ be a compact neighbourhood and $f, g\colon R^N\times V\to C$ be analytic functions of (t, z) in $W\times V$, where W is a certain neighbourhood of $t=0$. Suppose that, for all $z\in V$, $f(0,z)=0$, $\partial_i f(0,z)=0$ $(i=1,\ldots,N)$ and the matrix $H_{ij}(z)=-\partial_i\partial_j f(0,z)$ $(i, j=1,\ldots,N)$ satisfies $\mathrm{Re}\,H(z)\geqslant a>0$ (here ∂_i means $\partial/\partial t_i$). Suppose moreover that, with

$$D_n=\{t\in R^N|\,|t_i|<n^{-1/2}\lg n,\ i=1,\ldots,N\}, \tag{A3.1}$$

one has, for all $p>0$,

$$\lim_{n\to\infty} n^p \, . \max_{z\in V}\left(\int_{R^N\setminus D_n} |\exp(nf(t,z)+g(t,z))| \, \mathrm{d}t\right)=0. \tag{A3.2}$$

Then the sequence of analytic functions

$$\psi_n(z)=n^{N/2}\int_{R^N}\exp(nf(t,z)+g(t,z))\,\mathrm{d}t \tag{A3.3}$$

has an asymptotic series in powers of n^{-1} as $n\to\infty$, uniformly for $z\in V$.

Proof. Clearly, the asymptotic series does not change if the integration domain in (A3.3) is restricted to D_n. Therefore one has to consider

$$\int_{n^{1/2}D_n}\exp(nf(n^{-1/2}t,z)+g(n^{-1/2}t,z))\,\mathrm{d}t$$
$$=\exp(g(0,z))\int_{n^{1/2}D_n}\exp[-\tfrac{1}{2}(H(z)t,t)]\exp(n\tilde{f}(n^{-1/2}t,z)+g(n^{-1/2}t,z))\,\mathrm{d}t,$$

with the obvious definitions for $\tilde{f}(\tau,z)$, $\tilde{g}(\tau,z)$ which are analytic on $D_n\times V$ ($D_n\subset W$ for n sufficiently large), with power series in τ around $\tau=0$ starting with third- and first-order terms respectively. The derivative of the order k with respect to $n^{-1/2}$ of $n\tilde{f}(n^{-1/2}t,z)+\tilde{g}(n^{-1/2}t,z)$ is bounded uniformly on $(t,z)\in n^{1/2}D_n\times V$, by constant$(k)\,.\,(\lg n)^{k+2}$. Therefore the same derivative of $\exp(n\tilde{f}+\tilde{g})$ is bounded on the same domain by constant$(k)\,.\,(\lg n)^{3k}$, and the following Taylor expansion holds uniformly on $n^{1/2}D_n\times V$:

$$\exp(n\tilde{f}(n^{-1/2}t,z)+\tilde{g}(n^{-1/2}t,z))$$
$$=\sum_{l=0} Q_1(t,z)n^{-1/2}+\mathrm{O}\{[n^{-1/2}(\lg n)^3]^{k+1}\}$$

where Q_1 are polynomials in t, with $Q_1(-t,z)=(-1)^1Q_1(t,z)$. On the other hand,

$$\max_{z\in V}\left|\int_{R^N\setminus n^{1/2}D_n}\exp[-\tfrac{1}{2}(H(z)t,t)]Q_1(t,z)\,\mathrm{d}t\right|$$
$$\leqslant \text{constant}(l)\,.\,\exp[-\tfrac{1}{2}a(\lg n)^2].$$

Denoting

$$\varphi_l(z)=\exp(g(0,z))\int_{R^N}\exp[-\tfrac{1}{2}(H)(z)t,t)]Q_{21}(t,z)\,\mathrm{d}t \tag{A3.4}$$

and collecting the estimates, we shall have, for every integer $p\geqslant 0$,

$$\lim_{n\to\infty} n^p\max_{z\in V}\left|\phi_n(z)-\sum_{l=0}^{p}\varphi_l(z)n^{-1}\right|=0 \tag{A3.5}$$

which is the desired result.

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